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Letter to the Editor

Apparently first closed-form solutions of inhomogeneous circular plates in 200 years after Chladni

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1. Introduction

The first work dedicated to the free vibrations of circular isotropic plates was apparently due to Chladni [1] 200 years ago. The state of the art on this topic was summarized by Leissa [2]. Most papers are devoted to the axisymmetric vibrations. Kovalenko [3] was apparently the first who provided the exact solutions for non-axisymmetric vibrations for a single nodal diameter of a circular plate with linearly varying thickness. No closed-form solutions have been reported for plates with clamped or simply supported edges until Elishakoff [4] formulated the axisymmetric vibration frequencies of clamped plates by the semi-inverse method. Here, the apparently first closed-form solutions are reported for the circular plates in the *non-axisymmetric* setting.

2. Governing differential equation

In polar co-ordinates (r, θ), The equation governing the forced vibration of a circular plate with loading $q(r, \theta)$ and flexural rigidity $D(r)$ that varies with the polar radius r is [3]

$$
\begin{aligned}
&D\nabla^4 w + 2\frac{\mathrm{d}D}{\mathrm{d}r}\frac{\partial}{\partial r}\nabla^2 w + \nabla^2 D \nabla^2 w \\
&\quad - (1-\nu)\left[\frac{1}{r^2}\frac{\mathrm{d}^2 D}{\mathrm{d}r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\mathrm{d}^2 D}{\mathrm{d}r^2}\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\mathrm{d}D}{\mathrm{d}r}\frac{\partial^2 w}{\partial r^2}\right] + \delta\frac{\partial^2 w}{\partial t^2} = q(r, \theta),
\end{aligned}
\tag{1}
$$

where $D(r)$ and the Laplacian in polar co-ordinates, are defined, respectively, as

$$
D = \frac{Eh^3}{12(1-\nu^2)}, \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{2}
$$

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Here δ is the mass per unit area of the plate and h is the plate thickness both of which we assume to be a function only of r. The parameters E and v denote the modulus of elasticity and Poisson ratio, respectively.

Setting $q = 0$ and

$$w = \exp[\mathrm{i}(\omega t + n\theta)]W(r) \quad (n = 0, 1, 2, \ldots) \tag{3}$$

in Eq. (1), we obtain the following equation on the radial portion of the free vibration mode shape corresponding to the natural frequency ω:

$$\begin{aligned} &D\left[r^4\frac{\mathrm{d}^4W}{\mathrm{d}r^4} + 2r^3\frac{\mathrm{d}^3W}{\mathrm{d}r^3} - (2n^2+1)r^2\frac{\mathrm{d}^2W}{\mathrm{d}r^2} + (2n^2+1)r\frac{\mathrm{d}W}{\mathrm{d}r} + n^2(n^2-4)W\right] \\ &\quad + \frac{\mathrm{d}D}{\mathrm{d}r}\left[2r^4\frac{\mathrm{d}^3W}{\mathrm{d}r^3} + (v+2)r^3\frac{\mathrm{d}^2W}{\mathrm{d}r^2} - (2n^2+1)r^2\frac{\mathrm{d}W}{\mathrm{d}r} + 3n^2rW\right] \\ &\quad + \frac{\mathrm{d}^2D}{\mathrm{d}r^2}\left[r^4\frac{\mathrm{d}^2W}{\mathrm{d}r^2} + vr^3\frac{\mathrm{d}W}{\mathrm{d}r} - n^2vr^2W\right] - \omega^2r^4\delta W = 0. \end{aligned} \tag{4}$$

Note that when $n = 0$, we recover the equation in Ref. [4] for the axisymmetric case. As in Ref. [4], we pose to find a stiffness distribution $D(r)$ for a given distribution of density $\delta(r)$ and postulated mode shape all of which are polynomials. The results presented below are limited to considerations of uniform density and the first "angular mode" ($n = 1$) for the cases of clamped and simply supported boundary conditions. A closed form expression for the natural frequency is also given.

3. Clamped edge

If the plate has radius R, the mode shape W must satisfy the boundary conditions

$$W = \mathrm{d}W/\mathrm{d}r = 0 \quad \text{at } r = R. \tag{5}$$

To obtain a candidate mode shape, consider the static deflection of a *uniform* clamped plate under the loading $q = q_0\cos\theta$, where q_0 is constant. Solving Eq. (1) subject to the above boundary conditions, we find

$$w(r,\theta) = \frac{q_0}{90D}r(r-R)^2(2r+R)\cos\theta. \tag{6}$$

Thus we take $n = 1$ and postulate the following mode shape $W(r)$:

$$W(r) = r(r-R)^2(2r+R). \tag{7}$$

The problem proceeds as follows: find density and stiffness distributions so that Eq. (4) is identically satisfied. We observe that if $\delta(r)$ is taken as a polynomial of degree m, then $D(r)$ must be of degree $m+4$.

For a uniform density $\delta(r) = a_0$, where a_0 must be positive, setting

$$D(r) = \sum_{i=0}^{4} b_i r^i \tag{8}$$

we obtain the following set of five linear equations on the six unknowns $\{b_0, b_1, b_2, b_3, b_4, \omega^2\}$:

$$\begin{aligned}
&R(v+3)b_1 - 5b_0 = 0,\\
&48R(v+3)b_2 - 24(v+9)b_1 + R^3 a_0\omega^2 = 0,\\
&3R(v+3)b_3 - (2v+13)b_2 = 0,\\
&12(3v+17)b_3 - 48R(v+3)b_4 + Ra_0\omega^2 = 0,\\
&21(4v+21)b_4 - a_0\omega^2 = 0.
\end{aligned} \tag{9}$$

From the last equation we have

$$\omega^2 = 21(4v+21)b_4/a_0, \tag{10}$$

where b_4 is arbitrary but positive.

The remaining coefficients are given by

$$\begin{aligned}
b_0 &= \frac{(v+3)(24v^3 + 874v^2 + 8495v + 21795)R^4 b_4}{40(v+9)(2v+13)(3v+17)},\\
b_1 &= \frac{(24v^3 + 874v^2 + 8495v + 21795)R^3 b_4}{8(v+9)(2v+13)(3v+17)},\\
b_2 &= -\frac{9(v+3)(4v+33)R^2 b_4}{4(2v+13)(3v+17)},\\
b_3 &= -\frac{3(4v+33)Rb_4}{4(3v+17)}.
\end{aligned} \tag{11}$$

Fig. 1 depicts the stiffness $(D/R^4 b_4)$ for three values of the Poisson ratio v.

A second solution can be obtained by selecting as the candidate mode shape the static deflection of a *uniform* clamped plate under the loading $q = q_1(r/R)\cos\theta$ where q_1 is constant [5]. Solving Eq. (1) subject to the clamped boundary conditions, we obtain

$$w(r,\theta) = \frac{q_1 R^4}{192D}\left(\frac{r}{R}\right)\left(1 - \frac{r^2}{R^2}\right)^2 \cos\theta. \tag{12}$$

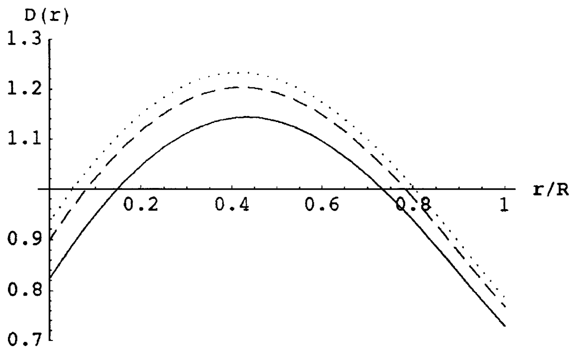

Fig. 1. Variation of the stiffness for a clamped circular plate corresponding to the mode shape $W = r(r-R)^2(2r+R)\cos\theta$: —, $v = 0$; - - -, $v = \frac{1}{3}$; …, $v = \frac{1}{2}$.

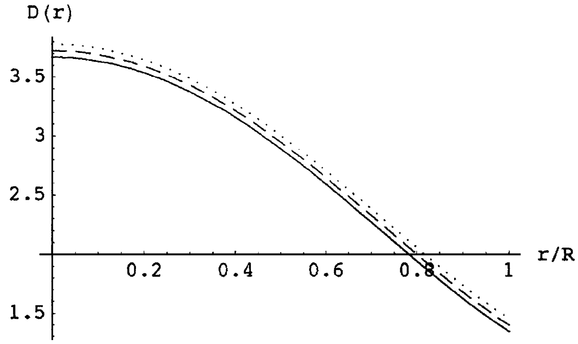

Fig. 2. Variation of the stiffness for a clamped circular plate corresponding to the mode shape $W = r(1 - r^2/R^2)^2 \cos\theta$: —, $v = 0$; - - -, $v = \frac{1}{2}$; …, $v = 1$.

We now take the radial portion of the above expression for $W(r)$ in Eq. (4) with $n = 1$ and again assume a constant density a_0 and undetermined stiffness $D(r) = \sum_{i=0}^{4} b_i r^i$. In this case, we obtain the following system of six homogeneous equations in the six unknowns $\{b_0, b_1, b_2, b_3, b_4, \omega^2\}$:

$$\begin{aligned} &b_1 = 0, \quad R^4 a_0 \omega^2 + 32[R^2(v+3)b_2 - 6b_0] = 0,\\ &3R^2(v+3)b_3 - (v+17)b_1 = 0,\\ &R^2 a_0 \omega^2 - 24[2R^2(v+3)b_4 - (v+11)b_2] = 0,\\ &b_3 = 0, \quad 128(v+8)b_4 - a_0\omega^2 = 0. \end{aligned} \tag{13}$$

Fortunately, the above system has a non-trivial solution. The natural frequency is given by

$$\omega^2 = \frac{128(v+8)}{a_0} b_4, \tag{14}$$

where b_4 is arbitrary but positive, while the remaining stiffness coefficients are given by

$$\begin{aligned} &b_0 = \tfrac{1}{9}R^4(v+33)b_4, \quad b_1 = 0,\\ &b_2 = -\tfrac{10}{3}R^2 b_4, \quad b_3 = 0. \end{aligned} \tag{15}$$

The corresponding plate stiffness can be expressed in the form

$$D(r) = \tfrac{1}{9}R^4[9(r/R)^4 - 30(r/R)^2 + 33 + v]b_4. \tag{16}$$

Fig. 2 depicts the stiffness $(D/R^4 b_4)$ for three values of the Poisson ratio v.

4. Simply supported edge

The mode shape W must now satisfy the boundary conditions

$$W = M_r = 0 \text{ at } r = R, \tag{17}$$

where the bending moment M_r is given by

$$M_r = -D\left[\frac{\partial^2 W}{\partial r^2} + v\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right]. \tag{18}$$

To obtain a candidate mode shape, consider the static deflection of a *uniform* simply supported plate under the loading $q = q_0 \cos\theta$ where q_0 is constant. Solving Eq. (1) subject to the above boundary conditions we obtain

$$w(r,\theta) = \frac{q_0 r[2(v+3)r^3 - 3R(v+4)r^2 + R^3(v+6)]}{90D(v+3)}\cos\theta. \tag{19}$$

Thus we take $n = 1$ and postulate the following mode shape $W(r)$:

$$W(r) = 2(v+3)r^4 - 3R(v+4)r^3 + R^3(v+6)r. \tag{20}$$

We proceed to find density and stiffness distributions so that Eq. (4) is identically satisfied.

For a uniform density $\delta(r) = a_0$, where a_0 must be positive. Taking $D(r)$ as in Eq. (8), we obtain the following set of five linear equations on the six unknowns $\{b_0, b_1, b_2, b_3, b_4, \omega^2\}$:

$$\begin{aligned}
&5b_0 - R(v+4)b_1 = 0,\\
&24(v+3)[(v+9)b_1 - 2R(v+4)b_2] - R^3(v+6)a_0\omega^2 = 0,\\
&(2v+13)b_2 - 3R(v+4)b_3 = 0,\\
&R(v+4)a_0\omega^2 + 12(v+3)[(3v+17)b_3 - 4R(v+4)b_4] = 0,\\
&21(4v+21)b_4 - a_0\omega^2 = 0.
\end{aligned} \tag{21}$$

From the last equation we have

$$\omega^2 = 21(4v+21)b_4/a_0, \tag{22}$$

where b_4 is arbitrary but positive.

The remaining coefficients are given by

$$\begin{aligned}
b_0 &= \frac{R^4(v+4)(24v^4 + 1018v^3 + 13307v^2 + 67761v + 118890)b_4}{40(v+3)(v+9)(2v+13)(3v+17)},\\
b_1 &= \frac{R^3(24v^4 + 1018v^3 + 13307v^2 + 67761v + 118890)b_4}{8(v+3)(v+9)(2v+13)(3v+17)},\\
b_2 &= -\frac{9R^2(v+4)^2(4v+33)b_4}{4(v+3)(2v+13)(3v+17)},\\
b_3 &= -\frac{3R(v+4)(4v+33)b_4}{4(v+3)(3v+17)}.
\end{aligned} \tag{23}$$

Fig. 3 depicts the stiffness $(D/R^4 b_4)$ for three values of the Poisson ratio v. Compared to the clamped plate, the stiffness is rather insensitive to v.

A second solution can be obtained by selecting as the candidate mode shape the static deflection $w(r,\theta)$ of a *uniform* simply supported plate under the loading $q = q_1 r/R \cos\theta$ where q_1 is constant [5]:

$$w(r,\theta) = \frac{q_1 R^4}{192D(3+v)}\left(\frac{r}{R}\right)\left(1 - \frac{r^2}{R^2}\right)\left[7 + v - (3+v)\left(\frac{r}{R}\right)^2\right]\cos\theta. \tag{24}$$

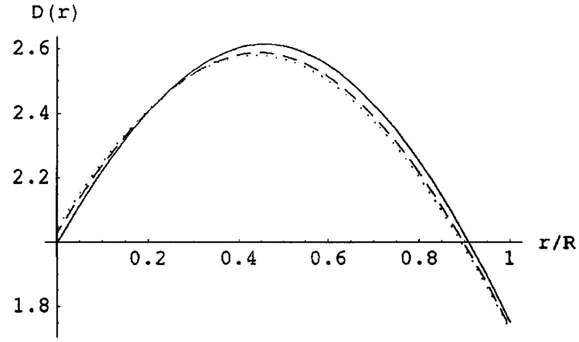

Fig. 3. Variation of the stiffness for a simply supported circular plate corresponding to the mode shape $W = [2(v+3)r^4 - 3R(v+4)r^3 + R^3(v+6)r]\cos\theta$: —, $v = 0$; - - -, $v = \frac{1}{3}$; …, $v = \frac{1}{2}$.

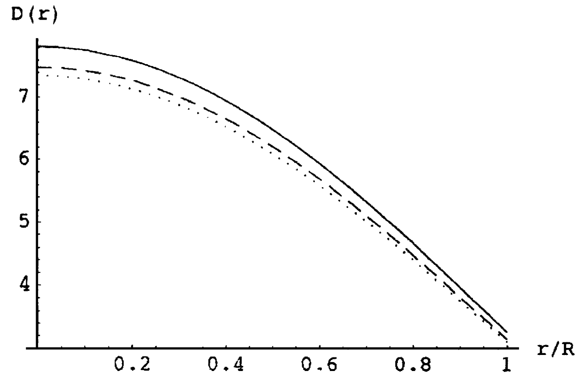

Fig. 4. Variation of the stiffness for a simply supported circular plate corresponding to the mode shape $W = r(R^2 - r^2)[7 + v - (3+v)r^2/R^2]\cos\theta$: —, $v = 0$; - - -, $v = \frac{1}{3}$; …, $v = \frac{1}{2}$.

Proceeding as above, we obtain the following system of six homogeneous equations in the six unknowns $\{b_0, b_1, b_2, b_3, b_4, \omega^2\}$:

$$\begin{aligned}
&b_1 = 0, \quad R^4 a_0(v+7)\omega^2 + 32(v+3)[R^2(v+5)b_2 - 6b_0] = 0,\\
&(v+17)b_1 - 3R^2(v+5)b_3 = 0,\\
&R^2 a_0(v+5)\omega^2 + 24(v+3)(v+11)b_2 - 48R^2(v+3)(v+5)b_4 = 0,\\
&b_3 = 0, \quad 128(v+8)b_4 - a_0\omega^2 = 0.
\end{aligned} \tag{25}$$

Fortunately, the above system has a non-trivial solution. The natural frequency is given by

$$\omega^2 = \frac{128(v+8)}{a_0} b_4, \tag{26}$$

where b_4 is arbitrary but positive, while the remaining stiffness coefficients are given by

$$b_0 = \frac{R^4(v^2 + 40v + 211)}{9(v+3)} b_4, \quad b_1 = 0,$$
$$b_2 = -\frac{10R^2(v+5)}{3(v+3)} b_4, \quad b_3 = 0. \tag{27}$$

The corresponding plate stiffness can be expressed in the form

$$D(r) = \frac{R^4[9(v+3)(r/R)^4 - 30(v+5)(r/R)^2 + v^2 + 40v + 211]}{9(v+3)} b_4. \tag{28}$$

Fig. 4 depicts the stiffness $(D/R^4 b_4)$ for three values of the Poisson ratio v.

5. Conclusion

It appears remarkable that whereas for the circular plate of *constant* flexural rigidity the mode shapes involve Bessel and trigonometric functions, simple polynomial and trigonometric functions are obtained under a flexural rigidity that varies. The unusual nature of this solution can be explained by our exploitation of the additional parameters arising due to inhomogeneity. The reported solutions can serve, for example, as benchmarks for verifying the accuracy of various numerical techniques.

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